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Abelian and Tauberian Theorems for a Class of Integral Transforms

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Extending the Wiener–Ganelius method we give Abelian and precise Tauberian remainder results for a class of Fourier kernels which includes the Hankel transform

$$f(x) = \int_0^\infty \sqrt{xu}\, J_\nu(xu)\, f(u)\, du, \qquad \nu \geqslant -\tfrac{1}{2}.$$

Further, we discuss applications to Fourier series and integrals.

1. INTRODUCTION

Recently Soni and Soni [9] have obtained Abelian and Tauberian theorems for a class of Fourier kernels $k(u)$ and monotone functions $f(u)$. Symmetric Fourier kernels give rise to a pair of reciprocal formulas on $L^2[0, \infty)$,

$$F(x) = \int_0^\infty k(xu)\, f(u)\, du, \tag{1.1}$$

$$f(u) = \int_0^\infty k(ux)\, F(x)\, dx. \tag{1.2}$$

As an important example we mention the Hankel kernel

$$k(u) = \sqrt{u}\, J_\nu(u), \qquad \nu \geqslant -\tfrac{1}{2},$$

which results in the sine and cosine kernels in the case of $\nu = \pm\frac{1}{2}$. In an earlier paper [6] we have proved a Tauberian remainder theorem for (1.1) connecting the asymptotic behavior of $f(u)$ at infinity $(u \to \infty)$ with $F(x)$ near the origin $(x \to 0+)$. Our object is to present a unified approach to Abelian and Tauberian remainder theorems which also includes the remaining case $u \to 0+$, $x \to \infty$. The Abelian results are well known and have

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been given by Aljančić *et al.* [1] and later in a slightly different form by Soni and Soni [9]. We obtain precise remainder theorems for general symmetric Fourier kernels $k(u)$ under suitable assumptions as in Soni and Soni [9] and functions $f(u)$ satisfying a general one-sided Tauberian condition. Our method is an important extension of Wiener's and Ganelius' theory, which works only for absolutely convergent integrals (1.1), L^1-kernels $k(u)$, L^∞-functions $f(u)$, and is not applicable in the case of conditional convergence. The same method also gives precise gap remainder theorems [6].

We begin with our basic assumptions and state for completeness the Abelian results in Section 2. Using these results and replacing $f(u)$ by the difference between $f(u)$ and its main term, we give in Section 3 the main Tauberian theorem in remainder form with remainders of polynomial decrease, followed by several conclusions.

For example, applying the Banach space-method of Ganelius [4] we can show that the remainder term of $f(u)$ cannot be substantially improved. In Remarks 4 and 5 we use ideas of Lyttkens [7]. Finally, Section 4 gives applications to Fourier integrals and generalized Fourier series, which were recently defined by Petrovich [8]. Further, we discuss convergence and summability of (multiple) Fourier series.

2. Basic Assumptions and Abelian Theorems

All functions are assumed to be real and measurable. For $f(u)$ we shall use the following conditions [5, 9]:

(a) $u^{\alpha_1} f(u) \in L^1(0, 1), \qquad \alpha_1 \geqslant 0,$

(b) $f(u) \in BV[1, \infty),$

(c) $f(u) \to 0 \quad \text{as} \quad u \to \infty,$

(d) $f(u) = O(u^{\tilde{\alpha}}), \qquad 0 < u \leqslant 1,\ \tilde{\alpha} \leqslant 0,$ (2.1)

(b) and (c) can be replaced by

(b*) $f(u) \in L^2[1, \infty)$ and $\int_0^\infty k(xu) f(u)\, du$ converges for $a \leqslant x < \infty$ $[0 < x \leqslant a]$,

(c*) $f(u) = O(u^{-\tilde{\alpha}}) \qquad \text{in} \quad 1 \leqslant u < \infty.$

Assumptions on the kernel:

(C_1) $k(u)$ and $k_1(u) := \int_0^u k(t)\, dt$ are bounded in $0 \leqslant u < \infty$.

(C_{1n}) $k(u) = \sum_{j=1}^n a_j \cdot u^{\alpha_j} + O(u^{\alpha_{n+1}}), \quad u \to 0+, \quad a_j \neq 0, \quad 0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$.

(C_2) $k(u) = O(u^{\alpha_1})$ as $u \to 0+$, $\alpha_1 > 0$, or $k(u) = k(0) + O(u^{\alpha_2})$ as

$u \to 0+$, $k(0) \neq 0$, $\alpha_2 > 0$, and $k_2(u) := \int_0^u k_1(t)\, t^{-1}\, dt$ is bounded in $0 \leqslant u < \infty$.

$k^{\mathrm{M}}(s)$ denotes the Mellin transform of $k(u)$,

$$k^{\mathrm{M}}(s) = \int_0^\infty u^{s-1} k(u)\, du,$$

and the integral converges absolutely or in the Cauchy sense.

LEMMA 1. *If $f(u)$ satisfies the conditions* (2.1)(a), (b), (c) *and $k(u)$ has the properties* (C_1) *and* (C_{10}), *then*

$$F(x) = \int_0^\infty k(xu)\, f(u)\, du, \qquad x > 0,$$

is well defined, and it holds that

$$F(x) = o(x^{-1}) \qquad as \quad x \to 0+,$$
$$F(x) = O(x^{\alpha_1}) \qquad as \quad x \to \infty.$$

$k^{\mathrm{M}}(s)$ is holomorphic in the strip $-\alpha_1 < \operatorname{Re} s < 1$. If $k(u)$ satisfies (C_{1n}), *then $k^{\mathrm{M}}(s)$ is meromorphic in $-\alpha_{n+1} < \operatorname{Re} s < 1$. The only singularities are simple poles at $s = -\alpha_j$ with residue a_j, $j = 1, 2, ..., n$. Also, $\int_0^\infty u^{-m}\{k(u) - \sum_{j=1}^n a_j u^{\alpha_j}\}\, du = k^{\mathrm{M}}(1-m)$, $-\alpha_{n+1} < 1 - m < -\alpha_n$.*

For the proof we refer to Soni and Soni [9, p. 171].

We need a further assumption:

(C_3) Let $k^{\mathrm{M}}(s)\, k^{\mathrm{M}}(1-s) = 1$, $0 < \operatorname{Re} s < 1$.

Then, $k(u)$ satisfying (C_1), (C_2), (C_3) is a symmetric Fourier kernel and (1.1) defines a unitary transformation on $L^2(0, \infty)$.

To give precise remainder results we have to study the growth of $1/k^{\mathrm{M}}(s)$ in a neighborhood of the imaginary axis.

The following condition (C_4) is satisfied by a large number of Fourier kernels including the Hankel transform where $k(u) = \sqrt{u}\, J_\nu(u)$, $\nu \geqslant -1/2$.

(C_4) $k^{\mathrm{M}}(s)$ is meromorphic and has no zeros in $-\alpha_1' < \operatorname{Re} s < 1 + \alpha_1$, $\alpha_1' > \alpha_1$, and it holds uniformly in the strip as $|\operatorname{Im} s| \to \infty$,

$$1/k^{\mathrm{M}}(s) = O((1 + |s|)^{-\operatorname{Re} s + 1/2}).$$

Our method is based on the Parseval relation given by Soni and Soni [9, p. 482]. We reformulate their result for our convenience:

LEMMA 2. *Assume that $f(u)$ and $k(u)$ satisfy the conditions* (2.1) *and* (C_1), (C_2), (C_3), *respectively. Furthermore,*

$$g(u) \in L^1(0, \infty) \cap BV[0, \infty), \qquad ug'(u) \in L^1(0, \infty),$$

$$\lim_{u \to \infty} g(u) = 0, \qquad G(x) = \int_0^\infty k(xu)\, g(u)\, du = o(x^{-\alpha_1 - \lambda}) \qquad \text{as } x \to \infty,\ \lambda > 1.$$

Then we have the validity of the Parseval relation

$$\int_0^\infty F(x)\, G(x)\, dx = \int_0^\infty f(u)\, g(u)\, du.$$

For a proof see [6, p. 16].

We now introduce two classes of auxiliary functions: The remainder of $F(x)$ will be dominated by $x^{m-1}L(x)\ \exp(-W(|\log x|/2\pi))(x \to \infty,\ x \to 0+)$ where $W(u)$ is positive, non-decreasing, subadditive and $L(u)$ slowly varying in the sense of Karamata, i.e., $L(u)$ positive, measurable, locally bounded and

$$L(\lambda u)/L(u) \to 1 \qquad \text{as} \quad u \to \infty \text{ for every } \lambda > 0.$$

Further, it is practical to assume that

$$L(u) = L(1/u), \qquad u > 0.$$

We shall make use of the following result due to Korevaar, Pitman, Soni and Soni [9, p. 173].

LEMMA 3. *Given a slowly varying function $L(u)$ and δ, A, B as positive constants, then there exist positive numbers C and Δ such that*

$$L(\lambda u)/L(\lambda) < Cu^{-\delta}, \qquad \lambda \geqslant \Delta,\ 0 < u \leqslant B,$$

and

$$L(\lambda u)/L(\lambda) < Cu^{\delta}, \qquad \lambda \geqslant \Delta,\ u \geqslant B.$$

If

$$f(u) = u^{-m}L(1/u), \qquad 0 < u \leqslant A,$$

then

$$f(u/\lambda)/f(1/\lambda) < Cu^{-m-\delta}, \qquad \lambda \geqslant \Delta,\ 0 < u \leqslant B,$$

and

$$f(u/\lambda)/f(1/\lambda) < Cu^{-m+\delta}, \qquad \lambda \geqslant \Delta,\ B \leqslant u \leqslant A\lambda.$$

Further

$$\lim_{u\to\infty} u^m L(u) \Big/ \int_{u_0}^{u} t^{m-1} L(t)\,dt = m,$$

$$\lim_{u\to\infty} u^{-m} L(u) \Big/ \int_{u}^{\infty} t^{-m-1} L(t)\,dt = m, \qquad m > 0.$$

We are now ready to formulate the Abelian and Tauberian theorems. The case $u \to \infty$, $x \to 0+$ was considered by Aljančić *et al.* [1] and later by Soni and Soni [9].

THEOREM 1 (Aljančić *et al.*). *Assume that $f(u)$ and $k(u)$ satisfy the conditions* (2.1)(a), (b), (c) *and* (C_1), (C_{10}), *respectively. Furthermore, for* $0 < u \leqslant 1$ [$1 \leqslant u < \infty$] *and* $0 < m < 1 + \alpha_1$,

$$f(u) = u^{-m} L(1/u)\,\phi(u) \qquad [u^{-m} L(u)\,\phi(u)],$$

$$L(u) \text{ slowly varying}, \qquad \phi(u) \in BV[0, \infty).$$

If $u^{-m}L(1/u)[u^{-m}L(u)]$ *is monotonic in* $0 < u \leqslant 1$ [$1 \leqslant u < \infty$] *or if* $\alpha_1 > 0$ *and* $m > 1$, *then*

$$F(x) = \int_0^\infty k(xu)\,f(u)\,du \sim \phi(0+)\,k^{\mathrm{M}}(1-m)\,x^{m-1}L(x) \qquad \text{as} \quad x \to \infty,$$

$$\sim [\phi(\infty)\,k^{\mathrm{M}}(1-m)\,x^{m-1}L(1/x) \text{ as } x \to 0+].$$

Proof. The proof follows from Theorems 1 and 4 in Aljančić *et al.* [1]. Nevertheless, we will give an outline.

We may suppose that $\phi(u)$ is positive non-increasing in $(0, \infty)$. Take the case $x \to \infty$. Notice that $\phi(u)\ L(1/u)$ is also a slowly varying function. We have

$$\frac{xF(x)}{x^m L(x)} = \left(\int_0^B + \int_B^x\right) k(u)\,u^{-m}\,\frac{L(x/u)}{L(x)}\,\phi(u/x)\,du + \int_x^\infty k(u)\,\frac{f(u/x)}{x^m L(x)}\,du$$

$$= I_1 + I_2 + I_3.$$

From (C_1) and (C_{10}) and for small $\delta > 0$ it follows that $\int_0^B |k(u)|\, u^{-m-\delta}\,du < \infty$. By Lemma 3 and the dominated convergence theorem

$$I_1 \to \phi(0+) \int_0^B u^{-m} k(u)\,du = \phi(0+)(k^{\mathrm{M}}(1-m) + O(B^{-m})).$$

If $u^{-m}L(1/u)$ is monotonic, by the second mean value theorem

$$I_2 = \phi(B/x)\, B^{-m} \frac{L(x/B)}{L(x)} \int_B^{\xi} k(u)\, du = O(B^{-m}) \qquad \text{as} \quad x \to \infty.$$

If $m > 1$, we apply the dominated convergence theorem since for small δ

$$\int_0^{\infty} |k(u)|\, u^{-m \pm \delta}\, du \text{ converges}, \qquad 1 < m \pm \delta < 1 + \alpha_1.$$

To estimate I_3 we write $f(u) = f_1(u) - f_2(u)$, where f_1 and f_2 are non-increasing and tend to zero as $u \to \infty$. Therefore

$$I_3 = O\left(\frac{f_1(1) + f_2(1)}{x^m L(x)}\right) \qquad \text{as} \quad x \to \infty.$$

THEOREM 2 (Aljančić *et al.*). *If $k(u)$ satisfies* (C_1) *and $f(u) = L(1/u)$ $[f(u) = L(u)]$, $f(u)$ non-increasing and tends to zero as $u \to \infty$, then*

(a)
$$\varliminf_{u \to \infty} k_1(u) \leqslant \varliminf_{\substack{x \to \infty \\ [x \to 0+]}} \frac{xF(x)}{L(x)} \left[\frac{xF(x)}{L(1/x)}\right] \leqslant \varlimsup_{\substack{x \to \infty \\ [x \to 0+]}} \frac{xF(x)}{L(x)} \left[\frac{xF(x)}{L(1/x)}\right] \leqslant \varlimsup_{u \to \infty} k_1(u).$$

(b) *If $\int_0^{\infty} u^{-1} k_1(u)\, du$ converges, then*

$$\int_0^x F(t)\, dt \sim L(x)[L(1/x)] \int_0^{\infty} k_1(u)\, u^{-1}\, du \qquad \textit{as} \quad x \to \infty [x \to 0+].$$

Proof. The proof employs ideas of Aljančić *et al.* [1, Theorem 3], who have given further interesting results of this kind. (a) By the second mean value theorem

$$\frac{xF(x)}{L(x)} = \int_0^B k(u) \frac{L(x/u)}{L(x)}\, du + \int_B^{\infty} k(u) \frac{L(x/u)}{L(x)}\, du$$
$$= \int_0^{\xi_x} k(u)\, du + o(1) \qquad \text{as} \quad x \to \infty,\ \xi_x \geqslant B.$$

For the other cases see Soni and Soni [9].

THEOREM 3. *If $f(u)$ and $k(u)$ satisfy the conditions of Theorem* 1 *with $m = 1 + \alpha_1$, and* (C_{11}), *then it follows that*

$$F_1(x) := \int_0^1 (k(u) - a_1 u^{\alpha_1}) f(u/x)\, du + \int_1^\infty k(u) f(u/x)\, du$$

$$\sim \begin{matrix} \phi(0+)\, x^m L(x) \\ [\phi(\infty)\, x^m L(1/x)] \end{matrix} \lim_{s \to -\alpha_1} \left\{ k^{\mathrm{M}}(s) - \frac{a_1}{s + \alpha_1} \right\} \quad \text{as} \quad \begin{matrix} x \to \infty \\ [x \to 0+] \end{matrix}.$$

Proof. The proof is similar to that of Theorem 1.

$$\frac{F_1(x)}{x^m L(x)} \to \phi(0+) \left(\int_0^1 u^{-m}(k(u) - a_1 u^{\alpha_1})\, du + \int_1^\infty k(u)\, u^{-m}\, du \right)$$

$$= \phi(0+) \lim_{s \to -\alpha_1} \left(k^{\mathrm{M}}(s) - \frac{a_1}{s + \alpha_1} \right) \quad \text{as} \quad x \to \infty.$$

In the case $x \to 0+$ we replace $L(x)$ by $L(1/x)$ and $\phi(0+)$ by $\phi(\infty)$.

THEOREM 4. *If $k(u)$ satisfies* (C_1) *and* (C_{1n}), $n \geqslant 1$, *and* $\alpha_n + 1 < m < \alpha_{n+1} + 1$, *then*

$$f(u) \sim u^{-m} L(1/u) [f(u) \sim u^{-m} L(u)] \quad \textit{as} \quad u \to 0+ \; [\textit{as } u \to \infty]$$

and

$$u^{\alpha_n} f(u) \in L^1(A, \infty) [u^{\alpha_{n+1}} f(u) \in L^1(0, A)]$$

for each $A > 0$ imply

$$F_2(x) := \int_0^\infty \left(k(u) - \sum_{j=1}^n a_j u^{\alpha_j} \right) f(u/x)\, du \sim f(1/x)\, k^{\mathrm{M}}(1 - m)$$
$$\textit{as} \quad x \to \infty \; [x \to 0+].$$

Proof. By (C_1), (C_{1n})

$$k(u) - \sum_{j=1}^n a_j u^{\alpha_j} = O(u^{\alpha_{n+1}}) \quad \text{as} \quad u \to 0+,$$
$$= O(u^{\alpha_n}) \quad \text{as} \quad u \to \infty.$$

From the dominated convergence theorem, Lemma 1 and Lemma 3 follows

$$\frac{F_2(x)}{f(1/x)} = \left(\int_0^{Ax} + \int_{Ax}^\infty \right) \left(k(u) - \sum_{j=1}^n a_j u^{\alpha_j} \right) \frac{f(u/x)}{f(1/x)}\, du$$
$$\to \int_0^\infty \left(k(u) - \sum_{j=1}^n a_j u^{\alpha_j} \right) u^{-m}\, du = k^{\mathrm{M}}(1 - m) \quad \text{as } x \to \infty \; [x \to 0+].$$

In a similar way one can prove Theorem 5:

THEOREM 5. *If $f(u)$ and $k(u)$ satisfy the conditions of Theorem* 4 *with $m = \alpha_n + 1$, then it follows that*

$$F_3(x) := \int_0^1 \left(k(u) - \sum_{j=1}^{n} a_j u^{\alpha_j}\right) f(u/x)\, du + \int_1^{\infty} \left(k(u) - \sum_{j=1}^{n-1} a_j u^{\alpha_j}\right) f(u/x)\, du$$

$$\sim f(1/x) \lim_{s \to -\alpha_n} \left\{ k^{\mathrm{M}}(s) - \frac{a_n}{s + \alpha_n} \right\} \qquad as \quad x \to \infty \ [x \to 0+].$$

3. A TAUBERIAN REMAINDER THEOREM

THEOREM 6. *Assume that $f(u)$ and $k(u)$ satisfy the conditions* (2.1) *and* (C_1), (C_2), (C_3), (C_4), *respectively. Furthermore,*

$$F(x) = \int_0^{\infty} k(xu) f(u)\, du = O\left(x^{m-1} L(x) \exp\left(-W\left(\frac{|\log x|}{2\pi}\right)\right)\right)$$

$$as \quad x \to \infty \ [x \to 0+], \tag{3.1}$$

$$F(x) \in L^1(0, 1) \qquad for \quad \alpha_1 = 0,\ m \geqslant 0,$$

$L(u)$ slowly varying, $L(u) = L(1/u)$, $W(u)$ positive, non-decreasing, subadditive such that

$$\lim_{u \to \infty} W(u)/u = 2\pi b \geqslant 0,$$

$$-\alpha_1 < m - b < 1 + \alpha_1 \ [-\alpha_1 < m + b < 1 + \alpha_1'].$$

If $f(u)$ satisfies the Tauberian condition

$$\sup(f(v) - f(u)) = O(u^{-m} L(u) R(u)), \tag{3.2}$$

$$u(1 - R(u)) \leqslant v \leqslant u \leqslant u_0 \ [u_0 \leqslant u \leqslant v \leqslant u(1 + R(u))],$$

where

$$R(u) = \exp\left(-\frac{W(|\log u|/(2\pi))}{m - b + 1 + \varepsilon}\right), \qquad m \geqslant b,$$

$$= \exp(-W(|\log u|/(2\pi))), \qquad m < b,$$

$$\left[R(u) = \exp\left(-\frac{W(|\log u|/(2\pi))}{m + b + 1 + \varepsilon}\right)\right], \quad for\ an\ \varepsilon > 0,$$

then

$$f(u) = O(u^{-m} L(u) R(u)) \qquad as \quad u \to 0+ \ [u \to \infty].$$

Proof. The proof is divided into six sections: First we transform the integral in (3.1) to convolution form

$$\psi(x) = K \times \phi(x).$$

Then we define a suitable function $Q(x)$ with

$$Q \times \psi(x) = (Q \times K) \times \phi(x) \qquad \text{for all real } x. \tag{3.3}$$

We insert Lemma 4 to prove that an important relation given by Ganelius [4] still holds:

$$\begin{aligned} |f(e(2\pi x))| &\leqslant 4 \sup_{\substack{0 \leqslant y - v \leqslant 2/\Omega \\ [v-y]}} \left(f(e(2\pi(x-y)))\, E(y) - f(e(2\pi(x-v)))\, E(v)\right) \\ &\quad + 6e(-2\pi x)\, |Q \times \psi(x)| =: T_1 + T_2, \end{aligned} \tag{3.4}$$

$$E(y) = \exp(-\xi y^2) =: e(-\xi y^2), \qquad \xi > 0.$$

Next, we estimate the first term T_1 on the right by aid of the Tauberian condition (3.2). To estimate the second term we use the assumption (3.1), and summarizing we find

$$\begin{aligned} |f(e(2\pi x))| &\leqslant K_2 e(-2\pi m x)\, L(e(2\pi x))\, R(e(2\pi x)) \\ &\quad + 8\Omega^{-1} \sup_{|v| < |x|/2} |f(e(2\pi(x-v)))\, E'(\tilde{y})|, \end{aligned} \tag{3.5}$$

$$v < \tilde{y} < y,\ x \leqslant x_0 < 0\ [y < \tilde{y} < v,\ x \geqslant x_0 \geqslant 0],\ \Omega = \mathrm{const}(R(e(2\pi x)))^{-1}.$$

Finally, iteration gives the theorem.

(1) We proceed as in [5, p. 596] and substitute

$$x \to e(-2\pi x), \qquad u \to e(2\pi y).$$

$$\psi(x) := F(e(-2\pi x)) = \int_{-\infty}^{\infty} K(x-y)\, \phi(y)\, dy = K \times \phi(x),$$

$$\phi(x) = e(2\pi x)\, f(e(2\pi x)), \qquad K(x) = 2\pi k(e(-2\pi x)).$$

$$\hat{K}(t) := \int_{-\infty}^{\infty} e(-2\pi i t x)\, K(x)\, dx = k^{\mathrm{M}}(it)$$

is meromorphic in $-1 - \alpha_1 < \operatorname{Im} t < \alpha_1'$, and there holds

$$g(t) := 1/\hat{K}(t) = O((1 + |t|)^{\operatorname{Im} t + 1/2}) \qquad \text{as} \quad |\operatorname{Re} t| \to \infty.$$

Furthermore,

$$\hat{K}(t)\, \hat{K}(-i - t) = 1, \qquad -1 < \operatorname{Im} t < \alpha_1.$$

We put

$$Q^F(t) := g(t) \int_{-\infty}^{\infty} \hat{E}^k(t-y)\, \hat{\chi}(y)\, e(-2\pi i\eta y)\, dy$$

and

$$E^k(x) = e(-\xi x^2 + 2\pi kx), \qquad \chi(x) = \Omega \sin^2(\pi\Omega x)/(\pi\Omega x)^2.$$

As in [5, p. 597] we find

$$Q(x) = 2\pi \int_{-\infty}^{\infty} H(x-y)\, e(2\pi y)\, k(e(2\pi y))\, dy, \qquad H(x) = E^k(x)\, \chi(x-\eta),$$

and

$$Q_1(x) := Q\left(\frac{\log x}{2\pi}\right)\Big/ x = \int_0^{\infty} k(xu)\, H\left(-\frac{\log u}{2\pi}\right) du.$$

$g(t)$ is holomorphic in the strip $-\alpha_1 - 1 < \operatorname{Im} t < \alpha_1'$. Hence

$$\begin{aligned} Q_1(u) &= O(u^{-\alpha_1'-1+\delta}) && \text{as} \quad u \to \infty, \\ &= O(u^{\max(0,\alpha_1-\delta)}) && \text{as} \quad u \to 0+ \text{ for every } \delta > 0. \end{aligned}$$

In a similar way we obtain$Q \times K(x) = H(x)$.

(2) By the Parseval relation and Lemma 2 we legitimate the change of order of integration for all real x as in [5, p. 597],

$$Q \times \psi(x) = H \times \phi(x).$$

By (3.1) and (3.2) we find

$$|\psi(x)| \leqslant c_1\, e(-2\pi(m-1)x)\, L(e(2\pi x))\, e(-W(|x|)), \qquad x \leqslant x_0 \ [x \geqslant x_0], \tag{3.6}$$

and

$$\sup_{\substack{[x_0/2 \leqslant x \leqslant y \leqslant x + c_2 R(e(2\pi x))] \\ x - c_2 R(e(2\pi x)) \leqslant y \leqslant x \leqslant x_0/2}} (f(e(2\pi y)) - f(e(2\pi x)))$$

$$= O(e(-2\pi mx)\, L(e(2\pi x))\, R(e(2\pi x))), \qquad x_0 \text{ suitably chosen.} \tag{3.7}$$

(3) By aid of the following Lemma 4 due to Ganelius [4, p. 19], we find the important relation (3.4).

LEMMA 4. *Let* $\phi(x) = e(2\pi x)\, f(e(2\pi x))$ *and* $f(u)$ *satisfy the conditions* (2.1). *Then, from* (3.3) *it follows for all real* x:

$$|\phi(x)| \leqslant 4 \begin{cases} -\inf\limits_{0 \leqslant v-y \leqslant 2/\Omega} \\ -\inf\limits_{0 \leqslant y-v \leqslant 2/\Omega} \\ \sup\limits_{0 \leqslant v-y \leqslant 2/\Omega} \\ \sup\limits_{0 \leqslant y-v \leqslant 2/\Omega} \end{cases} (\phi(x-y)\, E^k(y) - \phi(x-v)\, E^k(v)) + 6\,|Q \times \psi(x)|.$$

A proof is found in Ganelius [4] or in [6].

(4) To estimate the term T_1 in (3.4) we put $k=1$, $\xi = \pi$ and write

$$\begin{aligned} &\phi(x-y)\, E^1(y) - \phi(x-v)\, E^1(v) \\ &\quad = e(2\pi x)((f(e(2\pi(x-y))) - f(e(2\pi(x-v))))\, E(y) \\ &\qquad + f(e(2\pi(x-v)))(E(y) - E(v))). \end{aligned}$$

For $|v| \geqslant |x|/2$ both terms are $O(e(-\pi x^2/8))$ as $|x| \to \infty$. Now we choose

$$2(c_2 \Omega)^{-1} = R(e(2\pi x)). \tag{3.8}$$

In the case that $v \leqslant 0$ and $x \leqslant x_0$ $[v \geqslant 0,\ x \geqslant x_0]$, it follows from (3.7) and Lemma 3 that

$$\begin{aligned} &\sup_{\substack{0 \leqslant y-v \leqslant c_2 R(e(2\pi x)) \\ [v-y]}} (f(e(2\pi(x-y))) - f(e(2\pi(x-v))))\, E(y) \\ &\quad = O(e(-2\pi m x)\, L(e(2\pi x))\, R(e(2\pi x))). \end{aligned}$$

For $v > 0$ $[v < 0]$ we find the same estimate by an interval-splitting

$$x-y,\ x-y+t_1,\ x-y+t_2,\dots,\ x-y+t_n = x-v$$
$$[x-v,\ x-v+t_1,\ x-v+t_2,\dots,\ x-v+t_n = x-y],$$

observing that

$$e(\pm 2\pi v)\, E(y) n = O(1).$$

Thus, if $x \leqslant x_0$ $[x \geqslant x_0]$,

$$\begin{aligned} T_1 &\leqslant K_1 e(-2\pi m x)\, L(e(2\pi x))\, R(e(2\pi x)) \\ &\quad + 8\Omega^{-1} \sup_{|v| < |x|/2} |f(e(2\pi(x-v)))\, E'(\tilde{y})|, \qquad v < \tilde{y} < y \ [y < \tilde{y} < v]. \end{aligned} \tag{3.9}$$

(5) For the second term T_2 we proceed as in [4, pp. 34–40] and split the integral into the sum of integrals over the sets S_j,

$$\int_{-\infty}^{\infty} \psi(x-y)\, Q(y)\, dy = \left(\int_{-\infty}^{x-x_0[0]} + \int_{x-x_0[0]}^{0[x-x_0]} + \int_{0[x-x_0]}^{\infty} \right) \psi(x-y)\, Q(y)\, dy$$
$$=: I_1 + I_2 + I_3.$$

To estimate I_j we substitute $t \to t + i\delta_j$ belonging to the strip of holomorphy of $g(t)$, i.e., $-\alpha_1 - 1 < \delta_j < \alpha_1'$, $j = 1, 2, 3$. As in [4] we get

$$|I_j| \leqslant c_3 \int_{-\infty}^{\infty} \hat{E}(v) \int_{S_j} |\psi(x-y)\, e(-2\pi\delta_j y)$$
$$\times \int_{-\infty}^{\infty} e(2\pi i t(y-\eta))\, \hat{\chi}(t-v)\, g(t+i\delta_j)\, dt \,|\, dy\, dv.$$

By

$$\tilde{R}(x, y) := |\psi(x-y)|\, e(-2\pi\delta_j y), \qquad T(v, t) = \hat{\chi}(t-v)\, g(t+i\delta_j)$$

we have by Parseval relation

$$\begin{aligned} \|\hat{T}(v, \eta - y)\|_2^2 = \|T(v, t)\|_2^2 &= O((1 + |v| + \Omega)^{2+2\delta_j}), \qquad && 2 + 2\delta_j > 0, \\ &= O(1), && 2 + 2\delta_j < 0. \end{aligned} \tag{3.10}$$

Choosing

$$\begin{aligned} \delta_1 &= \min(m - b - 1 - \tilde{\varepsilon}, -1 - \alpha_1/2) \;\; [m + b - 1 - \tilde{\varepsilon}], \\ \delta_2 &= m - b - 1 - \tilde{\varepsilon} \;\; [m + b - 1 + \tilde{\varepsilon}], \qquad 0 < \tilde{\varepsilon} \leqslant \varepsilon, \\ \delta_3 &= m - b - 1 + \tilde{\varepsilon} \;\; [\max((\alpha_1 + \alpha_1')/2,\, m + b - 1 + \tilde{\varepsilon})], \end{aligned}$$

it follows by Lemma 3, $|x_0| \leqslant |x|$,

$$\|\tilde{R}(x, y)\|_2 = O(e(-2\pi(m-1)x)\, L(e(2\pi x))\, e(-W(|x|)))$$

and

$$\begin{aligned} |Q \times \psi(x)| &\leqslant c_4 \sum_{j=1}^{3} \int_{-\infty}^{\infty} e(-\pi v^2)\, \|\tilde{R}(x, y)\|_{2, S_j} \|T(v, t)\|_2\, dv \\ &\leqslant c_5\, e(-2\pi(m-1)x)\, L(e(2\pi x))\, R(e(2\pi x)). \end{aligned} \tag{3.11}$$

(3.11) also holds in the case $\alpha_1 = 0$.

(6) Adding the inequalities (3.9) and (3.11) we find (3.5), and by iteration as in [6, p. 31] we obtain the conclusion of the theorem.

Remark 1. Let the assumptions of Theorem 6 be satisfied. Further, assume that

$$e(W(u) - 2\pi bu) = O(1) \qquad \text{as} \quad u \to \infty,\ m - b > 0\ [m + b > 0],$$

and for every $\delta_0 > 1$ and $0 < 1/\delta_0 \leqslant \delta \leqslant \delta_0$,

$$L(u^\delta)/L(u) \leqslant K_{\delta_0}, \qquad u \geqslant u_0.$$

Then, the conclusion of Theorem 6 holds with $\varepsilon = 0$.

For the proof we must reinvestigate the estimate of the integral I_3 $[I_2]$. We set

$$I_{3[2]} = \left(\int_0^{\log \Omega/(2\pi)} + \int_{\log \Omega/(2\pi)}^{\infty[x - x_0]}\right) \psi(x - y)\, Q(y)\, dy =: I_a + I_b.$$

Taking

$$-1 < \delta_3 < m - b - 1 \quad [-1 < \delta_2 < m + b - 1] \qquad \text{in} \quad I_a$$

and

$$m - b - 1 < \tilde{\delta}_3 \qquad [m + b - 1 < \tilde{\delta}_3] \qquad \text{in} \quad I_b,$$

we have by aid of Lemma 3, as $x \to -\infty$ $[x \to \infty]$,

$$\begin{aligned}\|\tilde{R}(x, y)\|_{2,(0,\log \Omega/(2\pi))} = O(&\Omega^{m \mp b - 1 - \delta_{3[2]}}\, e(-2\pi(m - 1)x) \\ &\times L(e(2\pi x))\, e(-W(|x|))),\end{aligned}$$

and an analogous estimate for $\|\tilde{R}(x, y)\|_{2,(\log \Omega/(2\pi), \infty[x - x_0])}$ using $\tilde{\delta}_3[\tilde{\delta}_2]$. If $m = b$, $L \equiv 1$, it follows that

$$f(u) = O(\log(1/u)) \qquad \text{as} \quad u \to 0+.$$

Remark 2. Furthermore, we can formulate the "o"-version of Theorem 6 as follows: $f(u)$ and $k(u)$ satisfy the conditions (2.1) and (C_1), (C_2), (C_3), (C_4), respectively. Let

$$F(x) = o\left(x^{m-1} L(x)\, e\left(-W\left(\frac{|\log x|}{2\pi}\right)\right)\right) \qquad \text{as} \quad x \to \infty\ (x \to 0+),$$

$m, b, F(x)$, $L(u)$ and $W(u)$ as in Theorem 6. The Tauberian condition takes the form

$$\sup_{\substack{u(1 - \delta R(u)) \leqslant v \leqslant u \leqslant u_0(\delta) \\ [u_0(\delta) \leqslant u \leqslant v \leqslant u(1 + \delta R(u))]}} (f(v) - f(u)) \leqslant \omega(\delta)\, u^{-m} L(u)\, R(u),$$

where $\omega(\delta) \to 0$ as $\delta \to 0$. Then

$$f(u) = o(u^{-m}L(u)\,R(u)) \qquad \text{as} \quad u \to 0+ \ (u \to \infty).$$

Remark 3. Now we will show that our remainder results are in a certain sense best possible bounds. To do this we need two lemmas which can be found in a less general form in [4].

LEMMA 5. *If*

(i) *$k(u)$ satisfies the conditions* (C_1), (C_2), (C_3), (C_4),

(ii) $\hat{K}(t) = e(-2\pi i\sigma\zeta)\, d/d\sigma\, P(t, \zeta)$, $-1-\alpha_1 < \operatorname{Im} t < \alpha_1$, $\sigma = \operatorname{Re} t$, *and* $P(t, \zeta)$ *fulfills uniformly the inequality*

$$|P(t, \zeta)| \leqslant A(1 + |t|)^{-\operatorname{Im} t}, \qquad \operatorname{Re} t \geqslant \sigma_0,$$

where A is independent of ζ,

(iii) *every continuously differentiable $f(u)$ where $u^m f(u) = O(1)$,*

$$u^{m+1} f'(u) = O(1), \qquad 0 \leqslant u \leqslant \infty,$$

and

$$F(x) = O(x^{m-b-1}) \qquad \textit{as} \quad x \to \infty,\ m, b > 0,\ m - b > -\alpha_1,$$
$$[O(x^{m+b-1}) \qquad \textit{as} \quad x \to 0+] \qquad [m + b < \alpha_1 + 1]$$

also satisfies

$$f(e(2\pi x)) = O(e(-2\pi m x)/T(x)) \qquad \textit{as} \quad x \to -\infty \ [x \to \infty],$$

then

$$T(x) = O(|x|\, e(2\pi b\,|x|/(-b + m + 1))) \qquad \textit{as} \quad x \to -\infty,\ m > b,$$

$$T(x) = O(e(2\pi b\,|x|)),\ m \leqslant b,\ [T(x) = O(xe(2\pi b x/(b + m + 1)))\ \textit{as}\ x \to \infty].$$

Proof. Multiplying by $e(2\pi(m-1)x)$ we transform ϕ, K, ψ to $\tilde{\phi}$, $\tilde{K}$, $\tilde{\psi}$, respectively. Then

$$\tilde{\psi}(x) = \tilde{K} \times \tilde{\phi}(x) = O(e(-2\pi b\,|x|)) \qquad \text{as} \quad x \to \pm\infty,\ \tilde{\phi}, \tilde{\phi}' \in L^\infty(\mathbb{R}),$$
$$\tilde{\psi} \in L^\infty(-\infty, 0)[L^\infty(0, \infty)].$$

Put

$$\tilde{\phi}(x) = e(2\pi i\sigma(x - \omega) - \rho^2(x - \omega)^2/2), \qquad \rho > 1, \text{ and } \delta = \mp b.$$

Choosing

$$\rho = \rho_0 \sigma/(\log \sigma)^{1/2},$$

ρ_0 small enough, we get, as in [4, pp. 43–45],

$$\tilde{K} \times \tilde{\phi}(x) = O(e(\pm 2\pi b(x-\omega))\, \sigma^{\pm b - m} \log \sigma) \qquad (\sigma \to \infty).$$

Thus, by Ganelius' Banach space argumentation, we find

$$|T(\omega)| \leqslant K_1(\sigma + \sigma^{-m \pm b} \log \sigma \quad e(2\pi b\,|\omega|)), \qquad \omega \to -\infty,$$
$$[\omega \to \infty]$$

and the substitution

$$\sigma = e(2\pi b\,|\omega|/(m \mp b + 1))$$

gives the conclusion, $\sigma = \text{const}$, if $b \geqslant m$ as $\omega \to -\infty$.

LEMMA 6 (Landau). *For large τ independently of η,*

$$|I(\tau)| := \left| \int_\tau^\infty t^{-\gamma} e(it(\log t - \eta))\, dt \right| \leqslant C_\gamma \tau^{1/2-\gamma}, \qquad \gamma > 1/2,$$

$$|I(\tau)| := \left| \int_{\tau_0}^\tau t^{-\gamma} e(it(\log t - \eta))\, dt \right| \leqslant C_\gamma \tau^{1/2-\gamma}, \qquad \gamma \leqslant 1/2.$$

The proof is found in [4, 6].

THEOREM 7. *If $k(u)$ belongs to the Hardy–Titchmarsh class k' [10]:*

(i) $k^{\mathrm{M}}(s)\, k^{\mathrm{M}}(1-s) = 1$, $0 < \operatorname{Re} s < 1$,

(ii) *$k^{\mathrm{M}}(s)$ is holomorphic in the strip $\sigma_1 < \operatorname{Re} s < \sigma_2$, where $\sigma_1 < 0$, $\sigma_2 > 1$, except perhaps for a simple pole at $s = 0$,*

(iii) $k^{\mathrm{M}}(\bar{s}) = \overline{k^{\mathrm{M}}(s)}$,

(iv) $k^{\mathrm{M}}(s) = 2^{s-1/2}(\Gamma(s/2)/\Gamma(1/2 - s/2))(A_1 + A_2/s + O(1/|s|^2))$,
$\sigma_1 < \operatorname{Re} s < \sigma_2$, $\operatorname{Im} s \geqslant \tau_0$, *then the remainder of Theorem 6* $R(u) = u^{\pm b/(m \mp b + 1)}$, $m, b > 0$, $L(u) \equiv 1$, $W(u) = 2\pi b u$, $\sigma_1 < m - b < m + b < -\sigma_1 + 1$, *can be improved at most to*

$$\tilde{R}(u) = u^{\pm b/(m \mp b + 1)}/|\log u| \qquad as \quad u \to 0+ \ [u \to \infty],$$
$$\tilde{R}(u) = u^b, \qquad m \leqslant b,\ u \to 0+.$$

Proof. $k(u)$ satisfies (C_1), (C_2), (C_3), (C_4) [10]. Thus, we can apply Lemma 5 and Lemma 6 to obtain our result. By (iv) we find

$$k^M(\sigma + i\tau) = A\tau^{\sigma - 1/2} e(i(\tau \log \tau - \tau))(1 + a(\sigma)/\tau + O(1/\tau^2)), \qquad \tau \geqslant \tau_0,$$
$$\sigma_1 < \tau < \sigma_2.$$

Remark 4. If

(i) $f(u)$ and $k(u)$ satisfy the conditions (2.1) and (C_1), (C_{1n}), (C_2), (C_3), respectively, and $u^{\alpha_n} f(u) \in L^1(0, \infty)$,

(ii) $k^M(s)$ is meromorphic and has no zeros in $-\alpha_{n+1} < \operatorname{Re} s < 1 + \alpha_1$, and it holds uniformly in the strip as $|\operatorname{Im} s| \to \infty$ $1/k^M(s) = O((1 + |s|)^{-\operatorname{Re} s + 1/2})$, then

$$F(x) - \sum_{j=1}^{n} c_j x^{\alpha_j} \int_0^\infty u^{\alpha_j} f(u)\, du$$

$$= O\left(x^{m-1} L(1/x)\, e\left(-W\left(\frac{|\log x|}{2\pi}\right)\right)\right) \qquad \text{as} \quad x \to 0+,$$

$1 + \alpha_n < m + b < 1 + \alpha_{n-1}$ and (3.2) from Theorem 6 imply

$$f(u) = O(u^{-m} L(u)\, R(u)) \qquad \text{as} \quad u \to \infty.$$

Proof. We put

$$y_j = c_j \int_0^\infty u^{\alpha_j} f(u)\, du, \qquad j = 1, \ldots, n,$$

and define a function $g(u)$ where

$$y_j = a_j \int_0^\infty u^{\alpha_j} g(u)\, du \qquad \text{and} \qquad g(u) = 0, \qquad u \geqslant e(n-1).$$

By

$$g(u) := \sum_{j=1}^{n} d_j, \qquad 0 \leqslant u < 1,$$

and

$$g(u) := \sum_{j=v}^{n} d_j, \qquad e(v-2) \leqslant u < e(v-1),\ 2 \leqslant v \leqslant n,$$

we obtain the constants d_j by solving a system of linear equations with the non-vanishing Vandermonde's determinant:

$$(\alpha_j+1)\int_0^{e(n-1)} u^{\alpha_j} g(u)\,du = d_1 + d_2 e(\alpha_j+1) + \cdots + d_n e((n-1)(\alpha_j+1))$$

$$= (\alpha_j+1)\,y_j/a_j, \qquad j=1,\dots,n.$$

Finally, we can apply our Theorem 6 to

$$\int_0^{\infty} k(xu)(f(u)-g(u))\,du = O\left(x^{m-1}L(1/x)\, e\left(-W\left(\frac{|\log x|}{2\pi}\right)\right)\right)$$

$$\text{as} \quad x\to 0+,$$

which follows from the corresponding Abel result Theorem 4. A result for the case $x\to\infty$, $u\to 0+$ can be found in a similar way.

Remark 5. Assume that $f(u)$ and $k(u)$ satisfy the conditions (2.1) and (C_1), (C_{1n}), (C_2), (C_3), respectively. Furthermore, uniformly in the strip $-\alpha_1 < \operatorname{Re} s < 1+\alpha_{n+1}$ as $|\operatorname{Im} s|\to\infty$,

$$1/k^{M}(s) = O((1+|s|)^{-\operatorname{Re} s+1/2}).$$

Then, from

$$F(x) = O\left(x^{m-1}L(x)\, e\left(-W\left(\frac{|\log x|}{2\pi}\right)\right)\right)$$

$$\text{as} \quad x\to\infty, \quad -\alpha_{n+1} < m-b < -\alpha_n, \quad m > -\alpha_1,$$

and the Tauberian condition (3.2) it follows that

$$f(u) = \sum_{j=1}^{n} a_j u^{\alpha_j} F^{M}(\alpha_j+1) + O\left(u^{-m}L(1/u)\, e\left(-W\left(\frac{|\log u|}{2\pi}\right)\right)\right)$$

$$\text{as} \quad u\to 0+.$$

Proof. We proceed as in the proof of Theorem 6 and find

$$Q\times\psi(x) = \int_{-\infty}^{\infty} \psi(x-y) \sum_{k=1}^{n} 2\pi a_k \hat{H}(-i(1+\alpha_k))\, e(2\pi(\alpha_k+1)y)\,dy$$

$$+ \int_{-\infty}^{\infty} \psi(x-y)\, e(-2\pi\delta_j y)$$

$$\times \int_{-\infty}^{\infty} e(2\pi i t y)\; g(t+i\delta_j)\, \hat{H}(t+i\delta_j)\,dt\,dy,$$

$$-1-\alpha_{n+1} < \delta_j < -1-\alpha_n,$$

and the first term on the right is equal to

$$\int_{-\infty}^{\infty} H(x-y) \sum_{k=1}^{n} a_k e(2\pi(\alpha_k+1)y)\, F^{\mathrm{M}}(\alpha_k+1)\, dy.$$

However, in the case $m-b<0$, we can drop the Tauberian condition and directly use the inversion formula given by Hardy and Titchmarsh [10]: If $f(u) \in BV[a, \infty) \cap L^1(0, \infty)$ for each $a>0$ and $k(u)$ belongs to the class k', then

$$\tfrac{1}{2}[f(u+0)+f(u-0)] = \int_0^\infty k(ux)\, F(x)\, dx,$$

and we can apply an Abelian argument analogous to that of Theorem 1.

Remark 6. If $k(u)$ satisfies (C_1), (C_{11}), (C_2), (C_3), (C_4) and $k'(u) = a_1 \alpha_1 u^{\alpha_1 - 1} + O(u^{\alpha_2 - 1})$ as $u \to 0+$, then we can weaken the conditions (2.1) as follows: Let $f(u) \in BV[a, \infty)$ for each $a>0$ and $\lim_{u \to \infty} f(u) = 0$. If

$$G(x) = -\int_0^\infty k_1(xu)\, df(u)$$

converges for each $x>0$, then

$$f(u) = o(u^{-\alpha_1 - 1}) \qquad \text{as} \quad u \to 0+,$$

and

$$G(x)/x = \int_0^\infty k(xu)\, f(u)\, du = F(x).$$

Thus, Theorem 6 stays valid. However, the proof of the Parseval relation must be modified.

4. Applications

Applications to Fourier series, integrals and probability theory in the case $x \to 0+$, $u \to \infty$ are given in my paper [6].

(a) *Fourier series.* We state a corollary to Theorem 6: Given

$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx =: c(x) + s(x). \tag{4.1}$$

Suppose

$$\sum_{k=1}^{\infty} |a_k| + |b_k| < \infty.$$

Then from

$$c(x) - c(0) = O(x^{m+b} \log^{\rho}(1/x))[\text{resp. } o(\cdots)] \qquad \text{as} \quad x \to 0+,\ m \geqslant 0,\ b \geqslant 0,$$
$$(s(x) = O(x^{m+b} \log^{\rho}(1/x))[\text{resp. } o(\cdots)] \qquad \text{as} \quad x \to 0+,\ m > 0,\ b \geqslant 0)$$

and

$$a_n(b_n) \leqslant n^{-m-1} \log^{\rho} n \qquad \text{as} \quad n \geqslant n_0,\ \rho \in \mathbb{R},$$

it follows that

$$\sum_{k=n}^{\infty} a_k(b_k) = O(n^{-m} n^{-b/(m+b+1)} \log^{\rho} n)[\text{resp. } o(\cdots)] \qquad \text{as} \quad n \to \infty.$$

Let $g(u) \in L^1(-\pi, \pi)$ and $\phi_y(u) := g(y+u) + g(y-u) - 2g(y)$. For the partial sum $S_m(y)$ and the corresponding (C, 1) means $\sigma_m(y)$ of (4.1) we have

$$S_m(y) - g(y) = \frac{1}{\pi} \int_0^{\pi} \frac{\sin(m+1/2)u}{2 \sin(u/2)} \phi_y(u)\, du, \tag{4.2}$$

$$\sigma_m(y) - g(y) = \frac{1}{2\pi m} \int_0^{\pi} \left(\frac{\sin(mu/2)}{\sin(u/2)}\right)^2 \phi_y(u)\, du. \tag{4.3}$$

In the first case we put

$$f(u) = \phi_y(u)/(2\pi \sin(u/2)), \qquad 0 < u \leqslant \pi,$$
$$= 0, \qquad \pi < u,$$

and

$$F(x) = \int_0^{\infty} \sin(xu) f(u)\, du.$$

Obviously

$$F(m+1/2) - F(x) = O(1), \qquad \text{if} \quad g \in L^1(-\pi, \pi),$$
$$= O(1/m), \qquad \text{if} \quad g \in BV[-\pi, \pi], |x - m| < 1/2,$$

and our theorems apply to (4.2). In the second case we put

$$\begin{aligned} f(u) &= \phi_y(u)/(4\pi \sin^2(u/2)), && 0 < u \leqslant \pi, \\ &= 0, && u > \pi, \end{aligned}$$

and

$$F(m) := m(\sigma_m(y) - g(y)) = \int_0^\infty (1 - \cos(mu)) f(u)\, du.$$

Obviously, if $\phi_y(u) = O(1)$, $m < x < m+1$,

$$\begin{aligned} F(m) - F(x) &= 2\int_0^\pi \sin\left(\frac{m-x}{2}u\right)\sin\left(\frac{m+x}{2}u\right) f(u)\, du \\ &= O(m^\delta) \qquad \text{as} \quad m \to \infty, \delta > 0. \end{aligned}$$

This enables us to apply a Tauberian theorem to (4.3).

(b) *Multiple Fourier series.* Cheng [2] has considered the Riesz summation of multiple Fourier series

$$f(\mathbf{x}) \sim \sum a_{\mathbf{n}} e(i\mathbf{n} \cdot \mathbf{x}) \tag{4.4}$$

by spherical means. The spherical partial sum of the Fourier series (4.4) is roughly speaking the Hankel transform of the spherical mean function of $f(\mathbf{x})$. Thus, he is interested in Abelian and Tauberian $o(1)$-results and proves the following theorem: Let

$$\phi(t) \in L^1(0, a) \qquad \text{for each finite } a > 0,\ \phi(u) \geqslant 0,$$

and

$$u^{-q}\int_0^u |\phi(t)|\, dt \leqslant M, \qquad 0 < u < \infty,\ 0 < q < \mu + 1/2,\ \mu > 0.$$

Then

$$\Phi(x) := x^{q-\mu}\int_0^\infty J_\mu(xu)\, u^{-\mu}\phi(u)\, du \to 1qJ_\mu^{\mathrm{M}}(q-\mu) \qquad \text{as} \quad x \to \infty (x \to 0+),$$

implies

$$\lim_{\substack{u \to 0+ \\ (u \to \infty)}} u^{-q}\int_0^u \phi(t)\, dt = 1.$$

If $0 \leqslant q < \mu + 1/2$, we integrate by parts and find

$$\Phi(x) = x^{q+1/2-\mu} \int_0^\infty \sqrt{xu}\, J_{\mu+1}(xu)\, u^{-\mu-1/2} \times \int_0^u \phi(t)\, dt\, du =: x^{q+1/2-\mu} F(x).$$

Obviously, our Theorem 6 generalizes Cheng's result. An application to eigenvalue problems for elliptic differential operators is discussed in [4, 6].

(c) *Fourier integrals.* Let $k(u) = \cos u$, $f(u) = u^{-m} \int_0^u \sin(1/t^k)\, dt/t$, $0 < k/2 < m < k \leqslant 1$. It is easy to show that $f(u) = O(u^{-m+k})$ as $u \to 0+$. Integration by parts yields

$$xF(x) = m \int_0^\infty \sin(xu)\, u^{-m-1} \int_0^u \sin(1/t^k) \frac{dt}{t}\, du - \int_0^\infty \sin(xu)\, u^{-m-1} \sin(1/u^k)\, du =: I_1 + I_2.$$

Obviously,

$$I_1 = O(1), \qquad 0 \leqslant x < \infty.$$

To estimate I_2, we discuss

$$G_\pm(x) := \int_0^\infty \frac{e(\pm ixu - iu^{-k})}{u^{1+m}}\, du \qquad \text{as} \quad x \to \infty.$$

If we make the change of variables $u = x^{-1/(1+k)} v$, $s = x^{k/(1+k)}$, we obtain that

$$G_\pm = s^{m/k} \int_0^\infty \frac{e(is(\pm v - v^{-k}))}{v^{1+m}}\, dv =: s^{m/k} g_\pm(s).$$

Integrating by parts, we find

$$g_+(s) = O(1/s) \qquad \text{as} \quad s \to \infty,$$

and by the method of stationary phase [3, p. 31]

$$g_-(s) = O(1/\sqrt{s}) \qquad \text{as} \quad s \to \infty.$$

Thus,

$$F(x) = O(x^{(m-k/2)/(1+k)-1}) \qquad \text{as} \quad x \to \infty,$$

and Theorem 6 gives, as $u \to 0+$,

$$f(u) = O(u^{-m+b/(m-b+1)}), \qquad b = k\,\frac{1/2+m}{1+k}.$$

The best possible result is

$$f(u) = O(u^{-m+b/(m-b+1/2)}).$$

(d) *Generalized Fourier series.* Generalized Fourier series for 2π-periodic functions with at most one non-summable singularity at the point 0 were considered by Petrovich [8].

If for a real $\alpha > 0$,

$$|x|^{\alpha} f(x) \in L^1(-\pi, \pi)$$

and n is an integer such that

$$n - 1 < \alpha < n,$$

then Petrovich defines

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\cos(mu) - \sum_{j=0}^{[(n-1)/2]} (-1)^j \frac{(mu)^{2j}}{(2j)!}\right) f(u)\,du,$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\sin(mu) - \sum_{j=1}^{[n/2]} (-1)^{j+1} \frac{(mu)^{2j-1}}{(2j-1)!}\right) f(u)\,du$$

and

$$S(x) = \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Petrovich's results are purely of Abelian and Tauberian character. Our theorems clarify and generalize the situation.

If

$$f(u) = o(|u|^{-\alpha-1}) \qquad \text{as} \quad u \to 0,$$

then Theorem 4 gives

$$a_m, b_m = o(m^{\alpha}),$$

and one can show that $S(x)$ is summable by the (C, α) method almost everywhere to $f(x)$.

For $0 < \alpha < 1$ the converse statement holds: Let $S(x)$ be summable everywhere in $(0, 2\pi)$ by the (C, α) method to $f(x)$. If $f(u) - f(-u) + Cu^{-1-\beta}$

$|f(u)+f(-u)+C'u^{-1-\beta}|$ is monotonic in $0<u<u_0$, $\alpha \leqslant \beta < 1$, then it follows [8] that $u\tilde{f}(u) \in L^1[-\pi,\pi]$, $\tilde{f}(u) := f(u) \mp f(-u)$, and

$$F(m) = b_m = \frac{1}{\pi}\int_0^\pi (f(u)-f(-u))\,\sin(mu)\,du,$$

$$\left[F(m) = a_m = \frac{1}{\pi}\int_0^\pi (f(u)+f(-u))(\cos(mu)-1)\,du\right].$$

$F(m)$ and also $F(x)$ satisfy $F(x) = o(x^\alpha)$ as $x \to \infty$. Thus, by Theorem 6 and Remark 4,

$$f(u) \mp f(-u) = o(u^{-1-\beta}u^{(\beta-\alpha)/(\alpha+2)}) \qquad \text{as} \quad u \to 0+.$$

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